Pion Production in Pion-Pion Collisions*

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The strip-approximation procedure of Chew and Frautschi is applied in the pion-pion problem to estimate the cross section for the process $2\pi \rightarrow 4\pi$ considered in its full isotopic spin generality. The main results are dependent on S-wave scattering length formulas for low-energy pion-pion scattering. The inelastic cross sections are found to attain maxima not greater than 0.016 mb at total energies of about $11m_\pi$. Arguments based on the geometry of the double spectral functions suggest these results are accurate in the energy range from threshold at $4m_\pi$ to $8m_\pi$. On the basis of an estimation which is necessarily crude since the strip approximation breaks down in this case, the cross section for ρ production is found to go through a maximum of about 0.5 mb at an energy of $13m_\pi$, but ρ production is probably not significant at energies below $8m_\pi$. An attempt is also made to estimate quantitatively to what extent the four final pions emerge in two well-defined pairs and it is found that the effect is rather indistinct.

I. INTRODUCTION

It is the purpose of this note to attempt to calculate the cross section of the process $2\pi \rightarrow 4\pi$ on the basis of the strip approximation proposal of Chew and Frautschi.[1,2] While this reaction may not be experimentally feasible it is still of some theoretical interest to verify that the cross section is negligibly small in comparison to the elastic cross section throughout the low-energy resonance region as generally assumed in certain types of self-consistent calculations where the inelastic cross sections enter via the unitarity condition. Furthermore, due to the absence of spin complications, the π-π problem provides a relatively transparent demonstration of how the strip-approximation procedure yields information about isotopic spin branching ratios and angular distributions as well as total cross sections. In the following paragraph we briefly discuss the idea of the strip approximation in the case where the isotopic spin degree of freedom is suppressed. It is assumed the reader is familiar with the principles of the Mandelstam representation applied to the π-π problem.[3,4]

The elastic part of the double spectral function $\rho^{\text{el}}(s,t)$ is defined as that part of the entire double spectral function $\rho(s,t)$ which is obtained by writing the unitarity condition for diagrams that contain only elastic intermediate states in the s channel, where the arguments s and t are defined as the total energy squared in the corresponding channels. Then $\rho^{\text{el}}(t,s)$ is the contribution to $\rho(s,t)$ associated with diagrams containing only elastic intermediate states in the t channel. Now define the inelastic part of $\rho(s,t)$ by

$$\rho^{\text{in}}(s,t) = \rho(s,t) - \rho^{\text{el}}(s,t).$$

Since $\rho^{\text{el}}(s,t)$ vanishes in the strip defined by $4s/(s-16) \leq t \leq 16s/(s-4)$ we may apply crossing symmetry to obtain

$$\rho^{\text{in}}(s,t) = \rho^{\text{el}}(t,s) \tag{1}$$

in this strip. The strip approximation is then simply the statement that (1) holds for all values of s and t. Using the optical theorem and Eq. (1) we immediately obtain an expression for the inelastic cross section where s is the total energy squared;

$$\sigma(s) = 16s^{-1/2}(s-4)^{-1/2} \int dt\, \rho^{\text{el}}(t,s)[t^{-1} + (t+s-4)^{-1}]. \tag{2}$$

The term $(t+s-4)^{-1}$ may be neglected on the usual geometric grounds. The elastic double spectral function may be constructed from the crossed channel absorptive parts according to the Mandelstam prescription[3] by writing

$$\rho^{\text{el}}(s,t) \propto \int dt' \int dt'' J^*(t',s) J(t'',s) k^{-1/2}(s;t,t',t''), \tag{3}$$

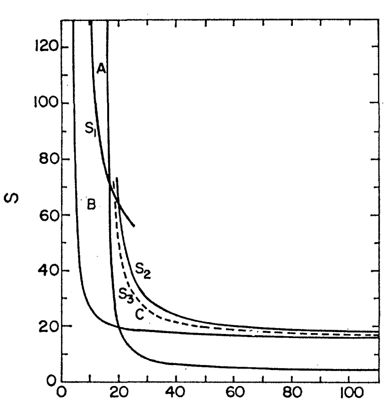

FIG. 1. The s-t plane showing boundaries of certain double spectral functions.

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[1] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **5**, 580 (1960).

[2] G. F. Chew and S. C. Frautschi, Phys. Rev. **123**, 1478 (1961).

[3] S. Mandelstam, Phys. Rev. **112**, 1344 (1958); **115**, 1752 (1959).

[4] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

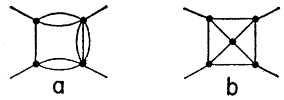

FIG. 2. Feynman diagrams having inelastic intermediate states in both channels.

where

$$k(s;t,t',t'')=t^2+t'^2+t''^2-2(tt'+tt''+t't'')-4tt't''/(s-4) \quad (4)$$

and the region of integration is for t', $t''\geqslant 4$ and $k>0$. The first variable in the absorptive part $J(t',s)$ is the square of the energy in the channel where J is the imaginary part while the second variable is the crossed channel energy squared. We now consider an expansion of the absorptive parts in polynomials in s and find how the region of convergence limits the accuracy of our results. The Legendre polynomial expansion

$$J(t',s)=\sum_l J_l(t')P_l(\cos\theta), \quad (5)$$

where $\cos\theta=1+2s/(t'-16)$, can converge only for $s<16t'/(t'-4)$ if $t'\leq 20$, or for $s<4t'/(t'-16)$ if $t'>20$, since otherwise the variable s runs into the region where the double spectral functions fail to vanish. However, from the limits on the integral, we see that only values of t' such that $k(s;t,t',4)>0$ need to be considered. So for a given t, the maximum value of s for which $\rho^{\text{el}}(s,t)$ is still constructed from a converging series is given by the smaller of s_1 and s_2, where these are the solutions of the equations

$$0=k[s_1;t,16s_1/(s_1-4),4]=k[s_2;t,4s_2/(s_2-16),4].$$

We find

$$s_1=(t-36)^{-1}(48+32t^{1/2}+4t) \quad (6)$$

$$s_2=16+192(t-16)^{-1}. \quad (7)$$

Figure 1 shows the inelastic strip (regions A and B) in the s,t plane. When the double spectral function is determined in this strip by applying the strip approximation to the elastic double spectral function (3), we can be certain that the polynomial expansion is convergent in regions B and C. Curves s_1 and s_2 correspond to the boundaries (6) and (7), respectively. The curve s_2 is, coincidentally, the boundary of a contribution to the true $\rho^{\text{in}}(s,t)$ that cannot be obtained by the strip approximation. This boundary arises from the singularity structure of the Feynman diagram[5] shown in Fig. 2(a). The only other such contribution intruding into region C comes from the diagram[5] shown in Fig. 2(b). The corresponding boundary is shown as the dashed line s_3 in Fig. 1. In fact, for $t\gg 16$ we have

$$s_2\sim 16+192t^{-1} \quad \text{and} \quad s_3\sim 16+128t^{-1}, \quad (8)$$

whereas the lower edge of the inelastic spectral function behaves like

$$16+64t^{-1}. \quad (9)$$

Hence, we are able to determine $\rho^{\text{in}}(s,t)$ by the polynomial expansion in the regions B and almost all of C. Although the strip is getting farther and farther from the physical region ($t<0$) as s goes to threshold at 16, so that the usual geometric arguments for the validity of the strip approximations do not seem to hold, we see from (8) and (9) that the range in t for which values of $\rho^{\text{in}}(s,t)$ are known becomes arbitrarily large. Hence, it is plausible that our results will be meaningful right down to threshold. This is in contrast to arguments based on the peripheral model in which one's ignorance about off-the-mass-shell extrapolations makes one wish to stay near the small momentum transfers that are possible only at high energies.[6,7]

In Sec. II we find the expressions corresponding to (2) for the π-π problem treated in its full isotopic spin generality and we also attempt to show how these expressions may be interpreted to obtain differential cross sections for pion production. Scattering length formulas describing low-energy elastic π-π scattering are introduced and the resulting numerical values of the total inelastic cross sections are presented in Sec. III. Also, we discuss a crude method for estimating the effect of $I=1$, $J=1$ resonances on the inelastic cross sections. Finally, in Sec. IV the angular distribution of the pion production is discussed in terms of the averages of certain angles chosen to illustrate the features of the one-pion exchange.

II. EXPRESSIONS FOR CROSS SECTIONS

The elastic unitarity condition for amplitudes of well-defined isotopic spin I ($I=0,1,2$) in the s channel can be written in the form

$$A_s^{\text{el}}(x,y,z;I)=(2/\pi)\int d^4r\,\delta(r^2-1)\delta(r'^2-1)\times A^*(x,y',z';I)A(x,y'',z'';I) \quad (10)$$

corresponding to the Cutkosky[8] diagram of Fig. 3. In accordance with Fig. 3 we have the definitions

$$x=(p+p')^2,\ y=(p-q)^2,\ y'=(p-r)^2,\ y''=(r-q)^2. \quad (11)$$

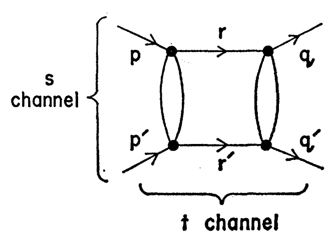

FIG. 3. Cutkosky diagram illustrating elastic unitarity in the s channel and inelastic unitarity in the t channel.

[5] The boundary curves for these and other diagrams in the pion-pion problem were found by V. A. Kolkunov, L. B. Okun, A. P. Rudik, and V. V. Sudakov, Zh. Eksperim. i Teor. Fiz. **39**, 340 (1960) [translation: Soviet Phys.—JETP **12**, 242 (1961)].

[6] S. D. Drell, Rev. Mod. Phys. **33**, 458 (1961).

[7] F. Salzman and G. Salzman, Phys. Rev. Letters **5**, 377 (1960); Phys. Rev. **125**, 1703 (1962).

[8] R. E. Cutkosky, J. Math. Phys. **1**, 429 (1960); Phys. Rev. Letters **4**, 624 (1960).

Only the positive energy roots of the arguments of the δ functions contribute to the integral. The notation associated with the amplitudes is the usual one; i.e., the first variable appearing in the argument of A is the value of the total energy squared in the s channel, the second and third are those of the crossed channels t and u, respectively, the subscript s denotes the absorptive part in the s channel, and the usual relation always holds among the energy variables;

$$x+y+z=4,\quad x+y'+z'=4,\quad x+y''+z''=4. \tag{12}$$

In addition, the subscript el reminds us that (10) is the unitarity condition for elastic intermediate states only.

Following the method introduced by Mandelstam,[3] we may now replace the amplitudes appearing in the integrand of (10) by one-dimensional dispersion relations that are integrals over crossed-channel absorptive parts. We can then obtain an expression for the elastic double spectral fiunction $A_{su}^{\mathrm{el}}(x,y,z;I)$ as an integral over products of the crossed-channel absorptive parts times the Mandelstam kernel by analytically continuing from negative physical momentum transfer z to positive z. All absorptive parts and elastic double spectral functions are thus far defined with respect to amplitudes of well-defined isotopic spin. If now we make the appropriate linear transformations on these expressions so that the absorptive parts and elastic double spectral functions correspond to the symmetric A, B, and C amplitudes constructed by Chew and Mandelstam,[4] the strip approximation takes the very concise form,

$$\begin{aligned} A_{su}^{\mathrm{el}}(x,y,z)&=B_{st}^{\mathrm{in}}(z,x,y),\\ B_{su}^{\mathrm{el}}(x,y,z)&=C_{st}^{\mathrm{in}}(z,x,y),\\ C_{su}^{\mathrm{el}}(x,y,z)&=A_{st}^{\mathrm{in}}(z,x,y). \end{aligned} \tag{13}$$

We make the substitutions (13) and undo the transformations to refer everything again to amplitudes of well-defined isotopic spin to obtain

$$A_{st}^{\mathrm{in}}(z,x,y;I)=\Gamma(j_I), \tag{14}$$

where

$$\begin{aligned} j_0&=\tfrac{1}{3}J_{00}+J_{11}+(5/3)J_{22},\\ j_1&=\tfrac{1}{8}J_{11}+\tfrac{5}{8}J_{22}+(5/12)J_{12}+\tfrac{1}{3}J_{01}+\text{c.c.},\\ j_2&=\tfrac{1}{8}J_{11}+(7/24)J_{22}+\tfrac{3}{4}J_{12}+\tfrac{1}{3}J_{02}+\text{c.c.},\\ J_{II'}&=A_s^*(y_1,x,z_1;I)A_s(y_2,x,z_2;I'), \end{aligned} \tag{15}$$

and we define the integral operator

$$\Gamma=(4/\pi)[x(x-4)]^{-1/2}\int dy_1dy_2[k(x;z,y_1,y_2)]^{-1/2}. \tag{16}$$

The region of integration is the usual one for the Mandelstam kernel, namely, $y_1\geqslant 4$, $y_2\geqslant 4$, and $k>0$. We want to preserve the association of y_1 and y_2 with particular vertices of Fig. 3 because in Sec. IV factors which distinguish between y_1 and y_2 are introduced into the integrand of (14). Hence, the notation c.c. is defined as the complex conjugate of the preceding expression with y_1 and y_2 interchanged.

The inelastic π-π cross sections can, in turn, be found by calculating the forward direction inelastic absorptive parts according to the well-known dispersion relation prescription and then using the optical theorem. We will use the expansion (5) and so absorptive parts will be assumed purely real. It seems valid to use only the S-wave term of (5) for low enough energies. The P-wave term is probably negligible until any $I=1$, $J=1$ resonances become important. It is actually impossible to include systematically a P-wave term in our approach since a logarithmically divergent integral is obtained as will be seen soon. The difficulty can be traced to the failure of the polynomial expansion to converge in the region of the double spectral function. Hence, we will assume the absorptive parts to be entirely S wave and, therefore, independent of momentum transfer.

The inelastic absorptive part is

$$A_s^{\mathrm{in}}(z,x',y;I)=\pi^{-1}\int_M^\infty dxA_{st}^{\mathrm{in}}(z,x,y;I)/(x-x'), \tag{17}$$

where the crossed-channel integral has been dropped as suggested after Eq. (2) and the lower limit is $M=4z/(z-16)$. Then, as in obtaining Eq. (2), the optical theorem leads to the inelastic cross section

$$\sigma^I(z)=16[z(z-4)]^{-1/2}\int_M^\infty x^{-1}dx\Gamma(j_I). \tag{18}$$

It is now possible to rearrange the order of integration in (18) to obtain

$$\sigma^I(z)=\left(\frac{64}{\pi}\right)[z(z-4)]^{-1/2}\int dy_1\int dy_2j_I \times\int_L^\infty dx[x^3(x-4)k(x;z,y_1,y_2)]^{-1/2}, \tag{19}$$

where

$$\begin{aligned} L=L(z,y_1,y_2)&=4+4zy_1y_2/G(z,y_1,y_2),\\ G(z,y_1,y_2)&=z^2+y_1^2+y_2^2-2(zy_1+zy_2+y_1y_2), \end{aligned}$$

and the y_1, y_2 region of integration is now for $4\leqslant y_1$, $4\leqslant y_2$, and $y_1^{1/2}+y_2^{1/2}\leqslant z^{1/2}$. Note that such a rearrangement would be impossible if j_I had an x dependency. Moreover, if j_I had a part that was linear in x (P-wave term), the integral over x would be logarithmically divergent as is easily seen.

The integral over x in (19) may be readily carried out this point but instead it proves enlightening to transform the integral into another form by the following considerations.[9] Cutkosky[8] has shown that the Mandel-

[9] Considerations similar to these were used by D. Amati, S. Fubini, A. Stanghellini, and M. Tonin, Nuovo Cimento 22, 569 (1961), to obtain a general model for high-energy peripheral collisions.

stam kernel is obtained by performing the four-dimensional integral corresponding to Fig. 3 with each energy denominator replaced by $2\pi i$ times a δ function. Thus, we have

$$2\pi^2[x(x-4)k(x;y,y_1,y_2)]^{-1/2} = (2\pi i)^2\int d^4r\,\delta(r^2-1)\delta(r'^2-1)\delta(y'-y_1)\delta(y''-y_2). \quad (20)$$

Consider the left side of (20) as the spectral function of a dispersion relation in the variable x for some real analytic function. Let us construct this dispersion integral and simultaneously apply the Cutkosky rule in reverse on the right-hand side of (20). Since the four-momenta r and r' are of the particles connecting the initial and final states in the channel where x is the energy squared, it is these four-momenta which go back into energy denominators. We obtain in this way

$$\int_{L'}^{\infty} dx'[x'(x'-4)k(x';y,y_1,y_2)]^{-1/2}(x'-x)^{-1} = \pi^{-1}\int d^4r\,\delta(y'-y_1)\delta(y''-y_2)(r^2-1)^{-1}(r'^2-1)^{-1}, \quad (21)$$

where $L'=L(y,y_1,y_2)$. Replacing y by z in the left-hand side of (21) as required in (19) is equivalent to interchanging Q and Q' of Fig. 3, so that we must also replace y'' by z'' in the right-hand side of (21) to maintain the equality. Making these replacements and setting $x=0$ gives the desired transformation.

The right-hand side of (21) with $x=0$ is what one would write down of the basis of the Cutkosky rule for generalized unitarity in the t channel and zero momentum transfer in the s channel for the diagram of Fig. 3. This diagram is the diffraction scattering associated with the inelastic cross section under consideration, r^2 is the momentum transfer squared in the inelastic process, while y_1 and y_2 are the total energies squared of the two pairs of final pions. We partially carry out the integration of the right side of (21) and substitute into (19) which then becomes

$$\sigma^I(z)=32[\pi z(z-4)]^{-1}\int dy_1\int dy_2 j_I \times\int_{L^-}^{L+} dr^2(r^2-1)^{-2}, \quad (22)$$

where

$$L^{\pm}=1+\tfrac{1}{2}(y_1+y_2-z)\pm\tfrac{1}{2}[G(z,y_1,y_2)(z-4)/z]^{1/2}.$$

The fact that r^2 has the physical significance of being the square of momentum transfer suggests that we may remove the integral signs and write the differential expression

$$\frac{d\sigma^I(z)}{dy_1dy_2d(r^2)}=\frac{32j_I}{\pi z(z-4)(r^2-1)^2}, \quad (23)$$

which is of the same form as the expression for the inelastic cross section derived by Salzman and Salzman[7] from the single-pion-exchange model. The advantage of our approach in deriving (23), already pointed out by several authors,[2,9] is that one has a better grasp of the approximations that enter the calculations as we have seen from the discussion of the strip geometry in Sec. I.

Note that the possibility of writing (23) in its differential form depended on the substitution (20) which introduced the differential dr^2 into our expression. The use of the Cutkosky rule essentially effected the transformation of the variable x to the variable r^2. The physical significance of x is rather obscure in that it runs over intermediate states of particles on their mass shells, whereas r^2 is clearly identified as the square of the four-vector of a virtual exchanged particle.

Returning to Eq. (22) we see that the r^2 integral is easily done yielding an expression for the total inelastic cross section

$$\sigma^I(z)=\frac{32}{\pi[z(z-4)]^{1/2}}\int dy_1\int dy_2\frac{j_I[G(z,y_1,y_2)]^{1/2}}{zy_1y_2+G(z,y_1,y_2)}. \quad (24)$$

In light of the above arguments which provide us with an interpretation of the variables r^2, y_1, and y_2 with respect to the inelastic process, let us inspect Eqs. (14) and (15). It is apparent that each term in (15) is the contribution of a particular combination of final isotopic spin states. Regardless of the techniques used in evaluating the integrals the numerical coefficients in (15), which are related to certain Clebsch-Gordan coefficients, survive and enable us to make some precise statements concerning relative production rates of pairs of well-defined isotopic spin. For a given combination of final isotopic spin states the relative rates of production from the three possible initial isotopic spin states are indicated in Table I.

TABLE I. Relative rates for producing a given final isotopic spin combination from the three possible initial isotopic spins. The proportions have significance only in vertical columns.

Initial isospin \ Final isospin combination	0,0	0,2	2,2	0,1	1,1	1,2
0	1	0	20	0	4	0
1	0	0	15	1	1	5
2	0	1	7	0	1	9

III. NUMERICAL VALUES

Scattering-length formulas for the absorptive parts of the $I=0$ and $I=2$ pion-pion S-wave scattering

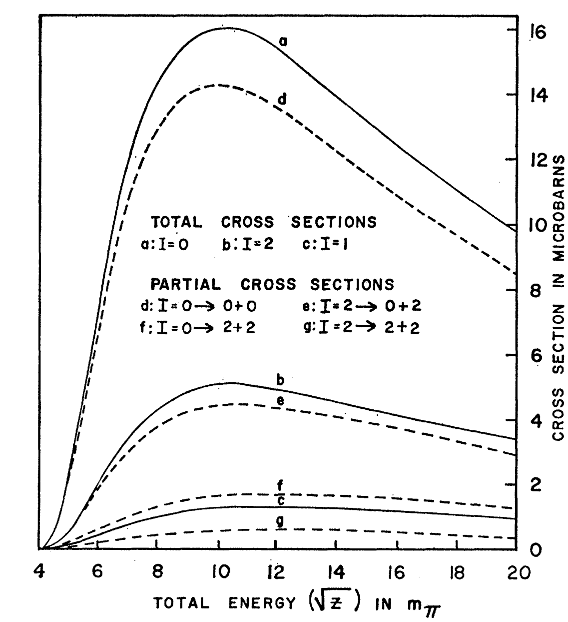

FIG. 4. Cross sections for $2\pi \rightarrow 4\pi$ where final pions are in two S-wave pairs. Solid curves are total cross sections while dashed curves are partial cross sections for production of the particular isotopic spin combinations indicated.

amplitudes are easily derived[10];

$$A_s(y;I)=\frac{a_I^2 f}{\{1+2(a_I/\pi)f\ln[\frac{1}{2}y^{1/2}(f+1)]\}^2+a_I^2f^2}, \tag{25}$$

where $f=(y-4)^{1/2}y^{-1/2}$, the scattering length of the state of isotopic spin I is a_I, and the x and z crossed-channel arguments have been explicitly dropped in the absorptive parts. The expressions $J_{II'}$ and j_I are then constructed according to (15) and substituted into (24).

The values of a_I selected for our use are from the analysis of Schnitzer[11]: $a_0=0.50m_\pi^{-2}$ and $a_2=0.16m_\pi^{-2}$. Schnitzer finds two sets of a_I but some independent evidence is given in his paper for preferring this set. The other set differs mainly in the sign of a_2 which would cause some enhancement of final $I=2$ pairs as seen from (25), but due to the smallness of a_2 the effect is not too significant.

Figure 4 shows the results of numerical integration of (24). Total inelastic cross sections are seen to go through maxima at approximately an energy of $11m_\pi$ and then fall off gradually. The maximum values are 0.016, 0.0014, and 0.0056 mb for isotopic spin equal to 0, 1, and 2, respectively. The behavior of the cross section just above threshold, as expected from phase-space consideration, is $\sigma\sim Q^{7/2}$ where Q is the total kinetic energy of the final pions.

It seems appropriate at this point to make some sort of estimate of the effect of the ρ in order to see where the results shown in Fig. 4 begin to leave us with an incomplete picture. The possibility of an $I=1$, $J=1$ resonance[12] below the mass of the ρ would subject the following remarks to some revision but in a qualitative sense they would remain essentially unchanged. To tackle the ρ let us crudely assume that Eq. (24) has been derived without neglecting the $I=1$ absorptive part but assuming that the absorptive part is independent of momentum transfer. The ρ, regarded as a pole at $m_\rho^2=28$ on the real axis of the energy plane, would appear in the absorptive part as a delta function of the form

$$A_s(y;1)=3\Gamma m_\rho(m_\rho^2-4)^{-1/2}\delta(y^{1/2}-m_\rho), \tag{26}$$

where Γ is the observed width of the ρ (about $0.72m_\pi$) and we have normalized the δ function so that the cross section attains the unitarity limit. We find, upon using (26) in (15) and neglecting the $I=0$ and $I=2$ absorptive parts, that production of two ρ's in an over-all $I=0$ state increases sharply from threshold at an energy of $10.6m_\pi$ and peaks at an energy $13m_\pi$ with a cross section about 30 times larger than the peak of the curve for $0\rightarrow 0{,}0$ in Fig. 4. Production of ρ's would actually take place below $10.6m_\pi$, at a threshold of $7.3m_\pi$, since it is unnecessary that both final pairs be $I=1$ as we have assumed here. By inspecting the combinations in (15) we see that the existence of the ρ enhances the partial cross sections for producing $I=0$ and 2 pairs in both the over-all $I=1$ and 2 cases, but in the over-all $I=0$ case the ρ's emerge as independent phenomena.

The partial cross sections (dashed lines) shown in Fig. 4 still stand unaffected by the existence of the ρ. The validity of these results is limited only by the considerations discussed in Sec. I. As it turns out, referring to Fig. 1, the boundary s_1 begins to cut into the inelastic strip at $s\approx 72$ corresponding to an energy of 8.5. The accuracy of the results of Fig. 4 for energies above this are thus dependent on how strongly the gradually narrowing strip B controls the amplitude in the nearby physical region. Hence, the picture is reasonably completely described by neglecting the ρ from energies of roughly $8m_\pi$ down to threshold. Above $10m_\pi$ and especially around $13m_\pi$, ρ's are expected to be the dominant feature of inelastic scattering, although the total inelastic cross sections still seem to remain quite small in comparison to typical elastic cross sections.

IV. ANGULAR FEATURES

In contrast to the statistical model, the single pion exchange or peripheral model predicts that final par-

[10] G. F. Chew, Lawrence Radiation Laboratory, Berkeley, suggested in a private communication that these expressions may be constructed by the N/D method where N is approximated by a_I and D is determined by the unitary condition.

[11] H. J. Schnitzer, Phys. Rev. **125**, 1059 (1962).

[12] Two resonances in the π^+-π^- state at 395 and 520 MeV were reported by N. P. Samios, A. H. Bachman, R. M. Lea, T. E. Kalogeropoulos, and W. D. Shephard, Phys. Rev. Letters **9**, 139 (1962). $I=2$ was ruled out. Also J. Schwartz, J. Kirz, R. Tripp, Bull. Am. Phys. Soc. **7**, 282 (1962), found an enhancement in the π^+-π^- cross section for mass >340 MeV. However, they suggest it occurs in the $I=0$ state.

ticles emerge from the region of interaction in two oppositely directed and fairly well-defined jets in the over-all barycentric system. Although not every event of the type $\pi+\pi\rightarrow 2\pi+2\pi$ will produce a clean separation of pion pairs, there is a tendency for such a separation to occur. Here we discuss two step-wise methods for estimating the extent of this separation on the basis of the expressions derived in Sec. II:

(1) One may think of the final pion pairs as being the decay products of particles of mass $y_1^{1/2}$ and $y_2^{1/2}$. As seen in the over-all barycentric system these unstable particles emerge at some angle θ, with respect to the direction of motion of the initial particles. It is then of interest to average θ by using (23) as a weighting factor and integrating over the physical ranges of the variables r^2, y_1, and y_2. We find

$$\cos\theta=\frac{2z^{1/2}[r^2-1+\frac{1}{2}(z-y_1-y_2)]}{[(z-4)G(z,y_1,y_2)]^{1/2}}.$$

The average angle $\bar{\theta}(z;I,I')$ for producing two pairs of isotopic spin I and I' at total energy $z^{1/2}$ is then obtained by replacing j_I by $J_{II'}$, carrying out the integration, and dividing by the appropriate normalizing factor.

(2) We then consider the angular distribution of the pions into which the unstable particle decays as observed in the overall barycentric system S_0. In keeping with the S-wave approximation the distribution of final momentum directions of the pair in the unstable particle rest frame S_r is spherically symmetric. Hence, the decays are uniformly distributed in $\cos\delta$ where δ is the angle measured in S_r between a pion momentum and the direction of the Lorentz transformation connecting S_0 to S_r. Letting ω be the angle in S_0 separating the pair produced by the decay of the particle of mass $y_1^{1/2}$, we find

$$\omega=\tan^{-1}[\sin\delta/(a+\gamma\cos\delta)]+\tan^{-1}[\sin\delta/(a-\gamma\cos\delta)],$$

where

$$a=[G(z,y_1,y_2)/4z(y_1-4)]^{1/2},$$
$$\gamma=(z+y_1-y_2)(4zy_1)^{-1/2}.$$

The average angle of separation, $\int_0^1\omega d(\cos\delta)$, is then inserted into the partially integrated expression (24). The overall average $\bar{\omega}(z,I;I')$ at total energy $z^{1/2}$ for the pair of isotopic spin I when the other pair has isotopic spin I' is finally obtained by replacing j_I by $J_{II'}$, carrying out the integration, and dividing by the appropriate normalizing factor.

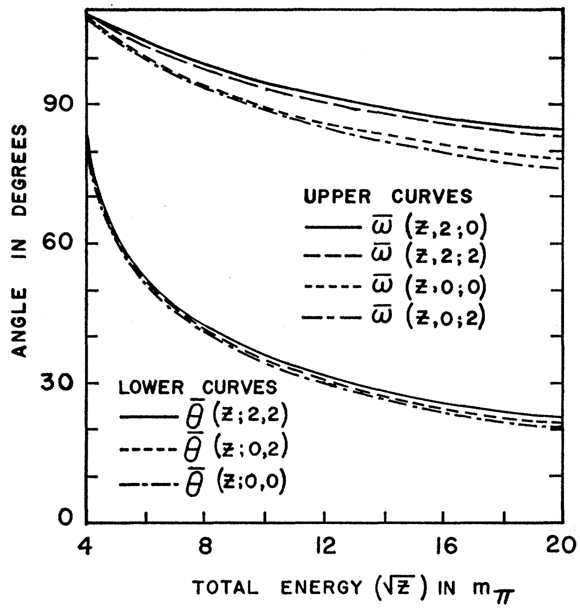

FIG. 5. Curves showing $\bar{\theta}(z;I,I')$, the average angle measured from the forward-backward direction for production of pairs of isotopic spin I and I', and $\bar{\omega}(z,I;I')$, the average opening angle of the isotopic spin I pair when the other pair has isotopic spin I'. The threshold values of $\bar{\theta}$ and $\bar{\omega}$ are 90° and the tetrahedral angle $\cos^{-1}(-\frac{1}{3})\approx 109.5°$, respectively.

Numerical values of these averages based on the scattering length formulas introduced previously are shown in Fig. 5. The emergence of final pairs in two jets is thus not a very pronounced feature in the energy range indicated and is especially poor in the range where our results are most meaningful. Although the smallness of $\bar{\theta}$ even for energies below $10m_\pi$ suggests that the unstable di-pions emerge fairly close to the forward and backward directions, the ensuing decay angle $\bar{\omega}$ is sufficiently large to wash out most of the effect.

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